Tree Recursion

Discussion 4: February 18, 2026

Definition: Tree recursive functions are functions that call themselves more than once.

Recursion takes practice. Please don't get discouraged if you're struggling to write recursive functions. Instead, every time you do solve one (even with help or in a group), make note of what you had to realize to make progress. Students improve through practice and reflection.

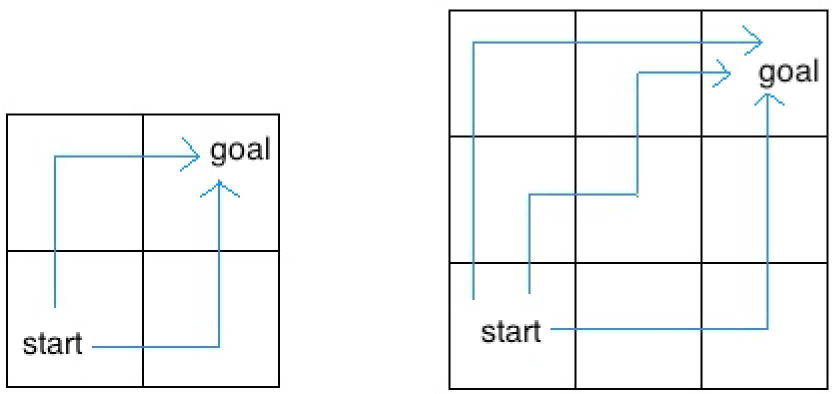
Tree Recursion

For the following questions, don't start trying to write code right away. Instead, start by describing the recursive case in words. Some examples:

- In `fib` from lecture, the recursive case is to add together the previous two Fibonacci numbers.
- In `double_eights` from lab, the recursive case is to check for double eights in the rest of the number.
- In `count_partitions` from lecture, the recursive case is to partition `n-m` using parts up to size `m` **and** to partition `n` using parts up to size `m-1`.

Q1: Insect Combinatorics

An insect is inside an `m` by `n` grid. The insect starts at the bottom-left corner `(1, 1)` and wants to end up at the top-right corner `(m, n)`. The insect can only move up or to the right. Write a function `paths` that takes the height and width of a grid and returns the number of paths the insect can take from the start to the end. (There is a closed-form solution to this problem, but try to answer it with recursion.)



Insect grids.

In the `2` by `2` grid, the insect has two paths from the start to the end. In the `3` by `3` grid, the insect has six paths (only three are shown above).

```
def paths(m, n):
    """Return the number of paths from one corner of an
    M by N grid to the opposite corner.

    >>> paths(2, 2)
    2
    >>> paths(5, 7)
    210
    >>> paths(117, 1)
    1
    >>> paths(1, 157)
    1
    """
    if m == 1 or n == 1:
        return 1
    return paths(m - 1, n) + paths(m, n - 1)
    # Base case: Look at the two visual examples given. Since the insect
    # can only move to the right or up, once it hits either the rightmost edge
    # or the upper edge, it has a single remaining path -- the insect has
    # no choice but to go straight up or straight right (respectively) at that point.
    # There is no way for it to backtrack by going left or down.
```

Note: This worksheet is a problem bank—most TAs will not cover all the problems in discussion section.

The recursive case is that there are paths from the square to the right through an (m, n-1) grid and paths from the square above through an (m-1, n) grid.

Presentation Time: Once your group has converged on a solution, it's time to practice your ability to describe why your recursive case is correct. Nominate someone and have them present to the group for practice. If you want feedback, ask.

Tree Recursion with Lists

Some of you already know list operations that we haven't covered yet, such as `append`. Don't use those today. All you need are list literals (e.g., `[1, 2, 3]`), item selection (e.g., `s[0]`), list addition (e.g., `[1] + [2, 3]`), `len` (e.g., `len(s)`), and slicing (e.g., `s[1:]`). Use those! There will be plenty of time for other list operations when we introduce them next week.

Important: The most important thing to remember about lists is that a non-empty list `s` can be split into its first element `s[0]` and the rest of the list `s[1:]`.

```
>>> s = [2, 3, 6, 4]
>>> s[0]
2
>>> s[1:]
[3, 6, 4]
```

Q2: Max Product

Implement `max_product`, which takes a list of numbers and returns the maximum product that can be formed by multiplying together non-consecutive elements of the list. Assume that all numbers in the input list are greater than or equal to 1.

```
def max_product(s):
    """Return the maximum product of non-consecutive elements of s.

    >>> max_product([10, 3, 1, 9, 2])   # 10 * 9
    90
    >>> max_product([5, 10, 5, 10, 5])  # 5 * 5 * 5
    125
    >>> max_product([])                 # The product of no numbers is 1
    1
    """
    if s == []:
        return 1
    if len(s) == 1:
        return s[0]
    else:
        return max(s[0] * max_product(s[2:]), max_product(s[1:]))
        # OR
        return max(s[0] * max_product(s[2:]), s[1] * max_product(s[3:]))
```

This solution begins with the idea that we either include `s[0]` in the product or not:

- If we include `s[0]`, we cannot include `s[1]`.
- If we don't include `s[0]`, we can include `s[1]`.

The recursive case is that we choose the larger of:

- multiplying `s[0]` by the `max_product` of `s[2:]` (skipping `s[1]`) OR
- just the `max_product` of `s[1:]` (skipping `s[0]`)

Here are some key ideas in translating this into code:

- The built-in `max` function can find the larger of two numbers, which in this case come from two recursive calls.
- In every case, `max_product` is called on a list of numbers and its return value is treated as a number.

An expression for this recursive case is:

```
max(s[0] * max_product(s[2:]), max_product(s[1:]))
```

Since this expression never refers to `s[1]`, and `s[2:]` evaluates to the empty list even for a one-element list `s`, the second base case (`len(s) == 1`) can be omitted if this recursive case is used.

The recursive solution above explores some options that we know in advance will not be the maximum, such as skipping both `s[0]` and `s[1]`. Alternatively, the recursive case could be that we choose the larger of:

- multiplying `s[0]` by the `max_product` of `s[2:]` (skipping `s[1]`) OR
- multiplying `s[1]` by the `max_product` of `s[3:]` (skipping `s[0]` and `s[2]`)

An expression for this recursive case is:

```
max(s[0] * max_product(s[2:]), s[1] * max_product(s[3:]))
```

Q3: Sum Fun

Implement `sums(n, m)`, which takes a total `n` and maximum `m`. It returns a list of all lists: 1. that sum to `n`, 2. that contain only positive numbers up to `m`, and 3. in which no two adjacent numbers are the same.

Two lists with the same numbers in a different order should both be returned.

Here's a recursive approach that matches the template below: build up the `result` list by building all lists that sum to `n` and start with `k`, for each `k` from 1 to `m`. For example, the result of `sums(5, 3)` is made up of three lists:

- `[[1, 3, 1]]` starts with 1,
- `[[2, 1, 2], [2, 3]]` start with 2, and
- `[[3, 2]]` starts with 3.

Hint: Use `[k] + s` for a number `k` and list `s` to build a list that starts with `k` and then has all the elements of `s`.

```
>>> k = 2
>>> s = [4, 3, 1]
>>> [k] + s
[2, 4, 3, 1]
```

```
def sums(n, m):
    """Return lists that sum to n containing positive numbers up to m that
    have no adjacent repeats.

    >>> sums(5, 1)
    []
    >>> sums(5, 2)
    [[2, 1, 2]]
    >>> sums(5, 3)
    [[1, 3, 1], [2, 1, 2], [2, 3], [3, 2]]
    >>> sums(5, 5)
    [[1, 3, 1], [1, 4], [2, 1, 2], [2, 3], [3, 2], [4, 1], [5]]
    >>> sums(6, 3)
    [[1, 2, 1, 2], [1, 2, 3], [1, 3, 2], [2, 1, 2, 1], [2, 1, 3], [2, 3, 1], [3, 1, 2],
    [3, 2, 1]]
    """
    if n < 0:
        return []
    if n == 0:
        sums_to_zero = []     # The only way to sum to zero using positives
        return [sums_to_zero] # Return a list of all the ways to sum to zero
    result = []
    for k in range(1, m + 1):
        result = result + [[k] + rest for rest in sums(n-k, m) if rest == [] or rest[0]
    != k]
    return result
```

The recursive case is that each list that sums to `n` is an integer `k` (up to `m`) followed by the elements of a list that sums to `n-k` and does not start with `k`.

Here are some key ideas in translating this into code:

- If `rest` is `[2, 3]`, then `[1] + rest` is `[1, 2, 3]`.
- In the expression `[... for rest in sums(...) if ...]`, `rest` will be bound to each of the lists within the list returned by the recursive call. For example, if `sums(3, 2)` was called, then `rest` would be bound to `[1, 2]` (and then later `[2, 1]`).

In the solution, the expresson below creates a list of lists that start with `k` and are followed by the elements of `rest`, first checking that `rest` does not also start with `k` (which would construct a list starting with two k's).

```
[[k] + rest for rest in sums(n-k, m) if rest == [] or rest[0] != k]
```

For example, in `sums(5, 2)` with `k` equal to 2, the recursive call `sums(3, 2)` would first assign `rest` to `[1, 2]`, and so `[k] + rest` would be `[2, 1, 2]`. Then, it would assign rest to `[2, 1]` which would be skipped by the `if`, avoiding `[2, 2, 1]`, which has two adjacent 2's.

You can use [recursion visualizer][RecursionVisualizerSumFun] to step through the call structure of `sums(5, 3)`.

If you get stuck (which many groups do), ask for help!

Optional Question

Tree recursion problems often appear on exams. Here's one:

Q4: A Perfect Question

This question was Fall 2023 Midterm 2 Question 4(a). The original exam version had an extra blank (where `total < k * k` appears below), but also included some guidance via multiple choice options and hints.

Definition. A *perfect square* is `k*k` for some integer `k`.

Implement `fit`, which takes positive integers `total` and `n`. It returns `True` or `False` indicating whether there are `n` **positive** perfect squares that sum to `total`. The perfect squares need not be unique.

```
def fit(total, n):
    """Return whether there are n positive perfect squares that sums to total.

    >>> [fit(4, 1), fit(4, 2), fit(4, 3), fit(4, 4)]  # 1*(2*2) for n=1; 4*(1*1) for n=4
    [True, False, False, True]
    >>> [fit(12, n) for n in range(3, 8)]  # 3*(2*2), 3*(1*1)+3*3, 4*(1*1)+2*(2*2)
    [True, True, False, True, False]
    >>> [fit(32, 2), fit(32, 3), fit(32, 4), fit(32, 5)] # 2*(4*4), 3*(1*1)+2*2+5*5
    [True, False, False, True]
    """
    def f(total, n, k):
        if total == 0 and n == 0:
            return True
        elif total < k * k:
            return False
        else:
            return f(total - k * k, n - 1, k) or f(total, n, k + 1)
    return f(total, n, 1)
```